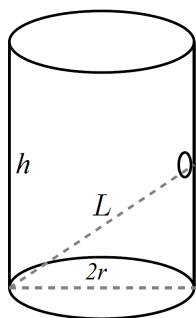


MATH 102:107, CLASS 16 (FRI OCT 13)

- (1) (Constrained optimization - Kepler's wedding) A cylindrical wine barrel has a hole in the center of one side. When a rod is put into this hole and reaches the furthest into the barrel that it can go, it reaches a distance of L . Given this constraint, find the radius r and height h which maximize the volume of the barrel.



Solution: Let r denote the radius of the barrel and let h denote the height.

$$\text{Constraint : } (2r)^2 + (h/2)^2 = L^2$$

$$\text{Objective function : } \pi r^2 h$$

We isolate r in the constraint equation:

$$4r^2 + h^2/4 = L^2 \implies r = \pm \sqrt{\frac{L^2}{4} - \frac{h^2}{16}}$$

and plug this expression for r back into the objective function to get it as a function of just the variable h .

$$f(h) = \pi r^2 h = \pi \left(\frac{L^2}{4} - \frac{h^2}{16} \right) h$$

$$f(h) = \frac{\pi L^2}{4} h - \frac{\pi}{16} h^3$$

In the situation of the problem, $h > 0$ (because height must be positive) and $h < 2L$ (because L is at least half the height) - therefore, we are looking for the global maximum of $f(h)$ in the interval $(0, 2L)$. We first find the critical points:

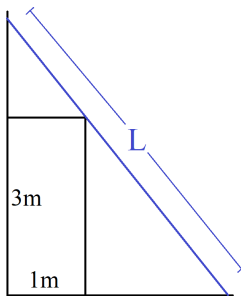
$$f'(h) = \frac{\pi L^2}{4} - \frac{3\pi}{16} h^2 = 0$$

$$\implies h^2 = \frac{4}{3} L^2 \implies \boxed{h = \frac{2}{\sqrt{3}} L}$$

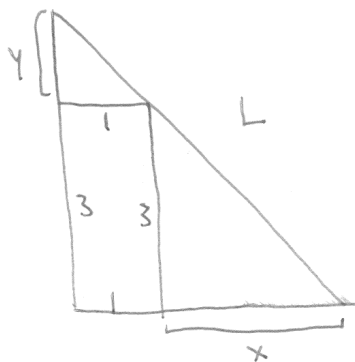
This is a local maximum because $f''(h) = -\frac{3\pi}{8}h$ which is negative at $h = \frac{2}{\sqrt{3}}L$. And it is a global maximum, because it's the only critical point. We can even check the endpoints of the interval:

$$f(0) = 0 \qquad f\left(\frac{2}{\sqrt{3}}L\right) = \frac{\pi}{3\sqrt{3}}L^3 \qquad f(2L) = 0$$

- (2) (Constrained optimization) A box of height 1m and depth 3m is placed against a wall. A straight ladder must go over the box and lean against the wall. What is the shortest possible length of the ladder?



Solution 1: Constrained optimization.



$$L^2 = (x+1)^2 + (y+3)^2 \leftarrow \text{Objective function}$$

$$\frac{y}{1} = \frac{3}{x} \leftarrow \text{Constraint}$$

Variables: x, y

Let x be the distance from the foot of the ladder to the box, and let y be the distance from the top of the box to the top of the ladder. Then our objective function, which we want to minimize, is¹

$$L^2 = (x+1)^2 + (y+3)^2$$

x and y can't vary freely, though - they are related. The constraint is that the ladder is a straight line - i.e., the right triangle formed by x and the height of the box, is similar to the right triangle formed by the width of the box and y . These similar triangles give us the constraint equation

$$y = \frac{3}{x}$$

¹Minimizing L is the same as minimizing L^2 , so we'll just do the second because it's easier.

Plugging this into the objective function gives us

$$\begin{aligned} f(x) &= L^2 = (x+1)^2 + \left(\frac{3}{x} + 3\right)^2 \\ &= (x+1)^2 + \left(\frac{3}{x}(x+1)\right)^2 = (x+1)^2 \left(1 + \frac{9}{x^2}\right) \end{aligned}$$

Take the derivative:

$$\begin{aligned} f'(x) &= 2(x+1) \left(1 + \frac{9}{x^2}\right) + (x+1)^2 \left(-\frac{18}{x^3}\right) \\ &= 2(x+1) \left(1 + \frac{9}{x^2} - (x+1)\frac{9}{x^3}\right) \\ &= 2(x+1) \left(1 - \frac{9}{x^3}\right) \end{aligned}$$

Remember that $x > 0$, because the foot of the ladder has to be to the right of the box. Therefore, the only critical point is $x = \sqrt[3]{9}$. To check that this is indeed a minimum, we have to take the second derivative

$$f''(x) = 2 \left(1 - \frac{9}{x^3}\right) + 2(x+1) \frac{27}{x^4} = 2 + \frac{36}{x^3} + \frac{54}{x^4}$$

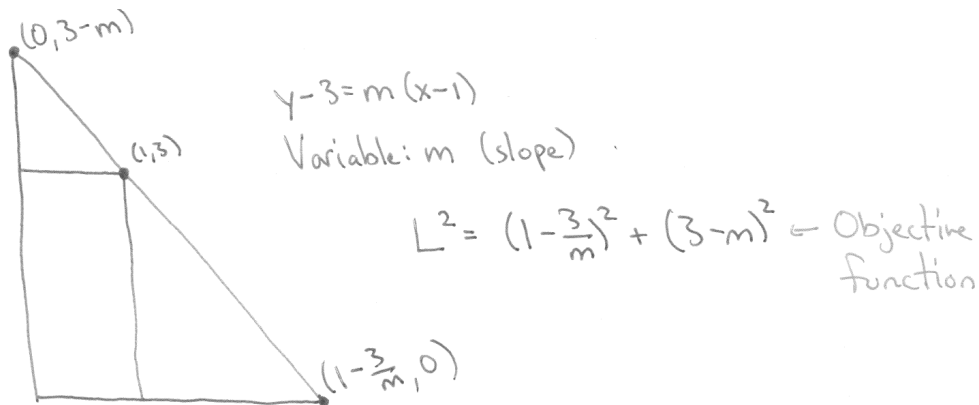
which is positive at $x = \sqrt[3]{9}$. Therefore, $f(x)$ is *concave up* at $x = \sqrt[3]{9}$, meaning that $x = \sqrt[3]{9}$ is indeed a local minimum.

To find the length of the ladder in this case, we can solve for y : $y = \frac{3}{\sqrt[3]{9}} = \sqrt[3]{3}$. Then we can use the Pythagorean theorem

$$\begin{aligned} L^2 &= (\sqrt[3]{9} + 1)^2 + (\sqrt[3]{3} + 3)^2 \\ &= \sqrt[3]{81} + 2\sqrt[3]{9} + 1 + \sqrt[3]{9} + 6\sqrt[3]{3} + 9 \\ &= 3\sqrt[3]{9} + 9\sqrt[3]{3} + 10 \end{aligned}$$

and so $L = \sqrt{3\sqrt[3]{9} + 9\sqrt[3]{3} + 10}$.

Solution 2: Unconstrained optimization.



We can put the picture in a coordinate system. The upper-right corner of the box is at the point $(1, 3)$, and the ladder is a line through this point with some (negative) slope m . Thus, the equation of the ladder is $y - 3 = m(x - 1)$. We can calculate the intersections of this line with the x -axis

$$-3 = m(x - 1) \implies x = 1 - \frac{3}{m}$$

and y -axis

$$y - 3 = m(-1) \implies y = 3 - m$$

Our objective function is thus

$$f(m) = L^2 = \left(1 - \frac{3}{m}\right)^2 + (3 - m)^2$$

which factors

$$\left(\frac{m-3}{m}\right)^2 + (m-3)^2 = (m-3)^2 \left(1 + \frac{1}{m^2}\right)$$

To minimize, we calculate the derivative

$$\begin{aligned} f'(m) &= 2(m-3) \left(1 + \frac{1}{m^2}\right) + (m-3)^2 \left(\frac{-2}{m^3}\right) \\ &= 2(m-3) \left(1 + \frac{1}{m^2} - \frac{1}{m^2} + \frac{3}{m^3}\right) = 2(m-3) \left(1 + \frac{3}{m^3}\right) \end{aligned}$$

Remember that m must be negative, because the foot of the ladder must be to the right of the box. Therefore, the only critical point is $m = -\sqrt[3]{3}$. To check this is a local minimum of $f(m)$, we can calculate the second derivative $f''(m)$ - this is similar to in Solution 1. We can also find the length of the ladder for this value of m , and will get the same answer as in Solution 1.

- (3) (Constrained optimization) *Baculovirus* is a cylindrically-shaped cell which must hold a certain amount of genetic material, and therefore has fixed volume $54000\pi nm^3$. Find the radius and height which give the cell the minimal possible surface area.

Solution: Our constraint is $\pi r^2 h = 54000\pi$, or equivalently, $r^2 h = 54000$. We must minimize the surface area, which is $2\pi r^2 + 2\pi r h$. We can isolate h in the constraint equation

$$h = \frac{54000}{r^2}$$

and plug it into the objective function to get

$$f(r) = 2\pi r^2 + 2\pi r \frac{54000}{r^2} = 2\pi r^2 + 2\pi \frac{54000}{r}$$

We are looking for a global minimum of this function on the interval $(0, \infty)$. Finding the critical points:

$$f'(r) = 2\pi \left(2r - \frac{54000}{r^2} \right) = 0$$

$$\implies r - \frac{27000}{r^2} = 0 \implies \boxed{r = 30}$$

This is a local minimum because $f''(r) = 2\pi(2 + \frac{108000}{r^3})$ is positive at $r = 30$. It's the only critical point, and therefore is a global minimum. (Checking the endpoints of the interval: as $r \rightarrow 0$, $f(r) \rightarrow \infty$ and as $r \rightarrow \infty$, $f(r) \rightarrow \infty$.)

- (4) (Unconstrained optimization) Let x measure the population of aphids in a garden. The reproduction rate of aphids is $G(x) = 3x$ and the rate of predation by ladybugs is $P(x) = \frac{30x}{5+x}$. Is there a value of $x > 0$ for which the net growth rate is *minimized*? At which it is *maximized*? For each, either find the value of x , or explain why none exists.

Solution: The net growth rate is

$$N(x) = G(x) - P(x) = 3x - \frac{30x}{5+x}$$

where x varies over the interval $[0, \infty)$. First, let's calculate the critical points of $N(x)$.

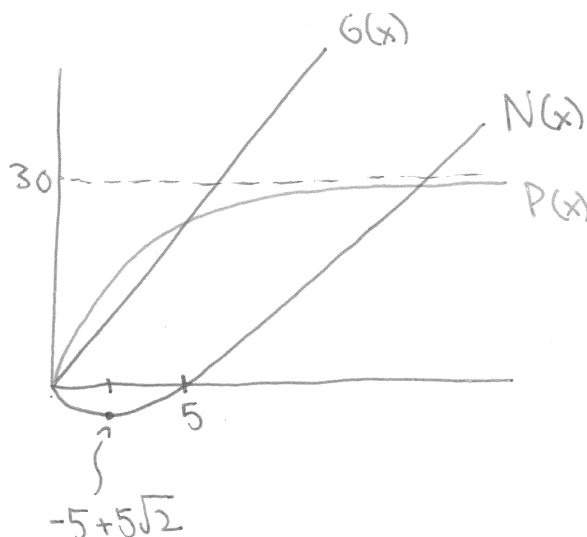
$$N'(x) = 3 - \frac{150}{(5+x)^2} = 0 \iff 3(5+x)^2 = 150$$

$$\iff 5+x = \pm 5\sqrt{2} \iff x = -5 \pm 5\sqrt{2}$$

We only need to consider the positive root, as the negative root lies outside of the range of the model. To classify this critical point, we calculate the second derivative

$$N''(x) = \frac{300}{(5+x)^3} \implies N''(-5+5\sqrt{2}) = \frac{300}{5^3} = \frac{12}{5}$$

which is positive - therefore, $x = -5 + 5\sqrt{2}$ is a local *minimum*. The derivative $N'(x) = 3 - \frac{150}{(5+x)^2}$ is negative for $0 < x < -5 + 5\sqrt{2}$, and positive for all $x > -5 + 5\sqrt{2}$. Therefore, $-5 + 5\sqrt{2}$ is a *global minimum*. The function has no global maximum, because as $x \rightarrow \infty$, $N(x) \rightarrow \infty$ (because $\frac{30x}{5+x}$ approaches towards the maximal rate of 30, while $3x$ grows without bound).



- (5) (Sketching) Let $f(x) = \frac{x^2}{2} - \frac{4}{x}$. Sketch a graph of $y = f(x)$. (Helpful steps: find the zeroes, find the critical points, classify them as local maxima/minima/neither, find the inflection points.)

Solution:

- For x near 0, $f(x) \sim -\frac{4}{x}$. In fact, $f(x)$ has a vertical asymptote at $x = 0$.
- As $x \rightarrow \pm\infty$, $f(x) \sim \frac{x^2}{2}$.

This gives us the first graph (left) below, as a starting point. The question is now, what is the intermediate behavior which occurs in between these two regimes. For example, how many times does the graph of $y = f(x)$ cross the x -axis, how many local minima/maxima are there, etc. We explicitly calculate the answer to these questions.

- (a) Zeroes of $f(x)$: $f(x) = 0 \iff \frac{x^2}{2} = \frac{4}{x} \iff x^3 = 8 \iff x = 2$. Therefore, there is only one zero, at $x = 2$.
- (b) Critical points of $f(x)$: $f'(x) = x + \frac{4}{x^2}$ and so $f'(x) = 0 \iff x^3 = -4 \iff x = -\sqrt[3]{4}$. Therefore, there is only one critical point, at $x = \sqrt[3]{4}$. One can reason out from the graph that it is a local minimum, but let's check the second derivative, just to be sure: $f''(x) = 1 - \frac{8}{x^3}$, and so $f''(-\sqrt[3]{4}) = 1 + \frac{8}{4} = 3$ is positive, indeed.
- (c) Inflection points: $f''(x) = 1 - \frac{8}{x^3}$ which is zero at just $x = 2$. This is an inflection point, because $f'''(x) = \frac{24}{x^4}$ is nonzero at that point.

The calculations above are recorded in the second graph (right) below.

